

Parameterizing the flattening of galaxies rotation curves on expanding locally anisotropic backgrounds

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Abstract

In this work are discussed possible generalizations of the expanding locally anisotropic metric ansatz with respect to approximately Newtonian many body gravitational systems. This ansatz describes local point-like matter distributions on the expanding Universe also allowing for a covariant parameterization of gravitational interactions at intermediate length scales. As an example of applicability it is modelled a disk galaxy model matching the physical parameters of the galaxy UGC2885 and it is shown that, by fine-tuning the metric functional parameter, the flattening of the galaxy rotation curve is fully parameterized by this metric. It is further analysed the mass-energy density corrections due to the expanding anisotropic background being shown that although there are negative contributions within the galaxy plane the total mass-energy density is strictly positive. Outside the galaxy plane, by considering an anisotropic functional parameter, it is shown that the mass-energy density is also strictly positive.

1 Introduction

In this work our main objective is to generalize to many body gravitational systems the recently suggested expanding locally anisotropic metric ansatz [1, 2] describing point-like local matter distributions in an expanding background, the Universe [3], as well as to show that such ansatz allows to parameterize corrections to the gravitational acceleration within intermediate gravitational scales. In particular, as an example, it is shown that for a specific fine-tuning of the functional parameter of this metric the flattening of galaxies rotation curves with respect to the predicted profiles, when considering only the Newtonian acceleration due to baryonic matter within galaxies [4, 5], is fully described such that this gravitational background ansatz can be interpreted as a covariant parameterization of the gravitational corrections attributed to cold dark matter [6]. Assuming a thin exponential disk approximation we apply such parameterization to a simplified lattice model roughly matching the characteristics of the galaxy UGC2885 [7–10].

The ELA metric ansatz was derived in [1] as a generalization of previous metric ansatze [11] describing local matter distributions in the expanding Universe [3] and noting that, although global space-time isotropy is mandatory, local matter distributions generate local anisotropy [12, 13]. Specifically the infinitesimal line element for this ansatz is

$$\begin{aligned}
 ds_{ELA}^2 &= (1 - U)(cdt)^2 \\
 &\quad - \frac{1}{1 - U} \left(dr_1 - \frac{H r_1}{c} (1 - U)^{\frac{\alpha}{2} + \frac{1}{2}} (cdt) \right)^2 - d\Omega^2 \\
 &= \left(1 - U - \left(\frac{H r}{c} \right)^2 (1 - U)^\alpha \right) (cdt)^2 \\
 &\quad + \frac{H r_1}{c} (1 - U)^{\frac{\alpha}{2} - \frac{1}{2}} dr_1 (cdt) - \frac{1}{1 - U} dr_1^2 - d\Omega^2
 \end{aligned} \tag{1}$$

where $U = 2GM/(c^2 r_1)$ is the Schwarzschild (SC) gravitational potential, $H = \dot{a}/a$ is the time dependent Hubble rate defined as the rate of variation of the Universe scale factor a , c is the speed of light in vacuum, M is the SC gravitational mass, G is the Gravitational constant, $d\Omega^2 = r_1^2(d\theta^2 + \sin^2 \theta d\varphi)$ is the solid angle line-element and we are employing spherical coordinates $(t, r_1, \varphi, \theta)$ for which the radial coordinate is an area radius (the area of a sphere is $A = 4\pi r_1^2$) such that the integration measure is independent of the Universe scale factor, $\sqrt{-g} = r_1^2 \sin(\theta)$. The ELA metric (1) interpolates between the SC metric [14] for small radii values (near the SC event horizon at the SC radius $r_1 \sim r_{1,SC} = 2GM/c^2$) and the Robertson-Walker metric [15] for large radii values ($r_1 \rightarrow \infty$). The shift function depends on a functional parameter α which, at the SC event horizon must be greater or equal to $\alpha(r_{1,SC}) \geq 3$ to prevent this horizon from being a space-time singularity, at the origin must have at least a leading divergence by $\alpha(r_1 \rightarrow 0) \sim -1/r_1$ to ensure that the SC mass

pole value coincides with the SC mass M and, at spatial infinity, it must be finite such that the shift function asymptotically vanishes and the RW metric is recovered [1].

As for the mass of the extended gravitational background encoded in the ELA metric in excess to the mass of the cosmological background described by the RW metric it was shown in [16] to be

$$\begin{aligned} M_\alpha &= \lim_{R_1 \rightarrow +\infty} 4\pi \int_0^{R_1} r_1^2 (\rho_{(\alpha)} - \rho_{RW}) dr_1 \\ &= \lim_{R_1 \rightarrow +\infty} \frac{H^2}{4G} \left(\left(1 - \frac{2GM}{c^2 R_1} \right)^{\alpha(R_1)} - 1 \right) R_1^3. \end{aligned} \quad (2)$$

In this expression $\rho_{(\alpha)}$ is the (extended) mass-energy contribution due to the expanding locally anisotropic metric in excess of the SC mass pole of value M and $\rho_{RW} = 3H^2/(8\pi G)$ is the cosmological background mass-energy density. The value for this mass is finite either when it is considered a radial cut-off for α above which this functional parameter is null such that the isotropic McVittie metric [11] is exactly recovered, or when the functional parameter is asymptotically null at spatial infinity being asymptotically proportional to $\sim \frac{1}{r^n}$ with $n \geq 2$. Hence either of these limits for α must be considered to ensure a finite mass contribution due to the expanding anisotropic background.

In addition, in between the two well established asymptotic limits (the SC metric and RW metric), there are no specific bounds on the functional parameter α such that it allows for a covariant parameterization of unmodelled observable effects at intermediate length scales. Also we recall that so far no direct physical interpretation for the functional parameter α exist, here the ELA metric ansatz is interpreted at most as a covariant parameterization of gravitational interactions corrections compatible with the well established asymptotic background solutions, the local SC metric and the RW metric. In this work we explore the application of such parameterization to describe the dynamics of many body gravitational systems, namely the flattening of rotation curves for galaxies, which indicates that such parameterization may be physically interpreted as cold dark matter. Such parameterization has also been applied to the Solar System dynamics [16], in particular allowing for a parameterization of the variation of the Astronomical Unit [17].

2 Ansätze for many body systems

With the objective of generalizing the ansatz (1) to many body gravitational systems let us note that, generally, the above ansatz for the ELA metric (1) is obtained from any given metric $g_{\mu\nu}$ describing the gravitational background of local matter distributions by considering the radial shift function $N_{ELA}^r = -H r_1 (\sqrt{g_{00}})^{\alpha+1}/c$. Hence to generalize the ELA metric ansatz to many body backgrounds it is first require to define the many body SC metric background. However no explicit analytical solution for a metric describing such background

is known. The standard approach is to consider a perturbative post-Newtonian solution to a specific order in the gravitational field U [18]. Here we will consider the Newtonian approximation which is commonly employed in macroscopical galaxy models [19] (see also [20]), hence we define the local Newtonian gravitational background for N massive bodies to first order in the gravitational field

$$ds_{SC.N}^2 = (1 - U_N)(cdt)^2 - dr^2 - d\Omega^2 ,$$

$$U_N = \sum_{n=1}^N U_n \quad , \quad U_n = \frac{2GM_n}{c^2|\mathbf{r}_1 - \mathbf{r}_{1.n}|} , \quad (3)$$

where U_N is the total SC gravitational potential being U_n the SC gravitational potential for each of the many bodies. The distance to each body position is given as usual by the Euclidean distance between the point at which the metric is being evaluated (\mathbf{r}_1) and the position of each massive point-like body ($\mathbf{r}_{1.n}$). We recall that for simulations within the Solar System the gravitational field is at most of order $U < 10^{-8}$ and a second order expansion on the gravitational field is commonly employed, hence being obtained an accuracy on the metric definition of order $\sim 10^{-16}$ which is enough to match the current experimental accuracy for observational data [21]. In this work we will consider a galaxy model based in an analytical thin disk approximation for which the SC gravitational field is typically of order $U_n < 10^{-7}$ such that the second order corrections would be at most of order $\sim 10^{-14}$. Hence the first order expansion considered for $g_{00.N}$ and $g_{rr.N}$ is accurate to order $\sim 10^{-7}$ being enough for exemplification purposes and to obtain a macroscopical estimative for the functional parameter of the ELA metric ansatz. A more detailed model would require a higher order expansion on the gravitational field.

Given the many body perturbative background (3) it is straight forward to generalize the ELA metric ansatz by considering the definition of the shift function with respect to the total gravitational potential

$$N_{ELA.\text{eff}}^r = -\frac{Hr}{c}(1 - U_N)^{\frac{\alpha_{\text{eff}}}{2} + \frac{1}{2}}$$

$$ds_{ELA.\text{eff}}^2 = (1 - U_N)(cdt)^2 - d\Omega^2$$

$$- (dr_1 + N_{ELA.\text{eff}}^r(cdt))^2 . \quad (4)$$

This metric has the same properties of the ELA metric (1), in particular at the event horizon for each of the n bodies there is no space-time singularity. We remark that the functional parameter α_{eff} parameterizes a collective gravitational correction to the SC metric due to the many bodies in the gravitational system such that a direct fit to the galaxy rotation curve can be carried allowing to map this functional parameter within the galaxy plane, hence to compute the mass-energy density and anisotropic pressures within the galaxy.

To further compute the mass-energy density and anisotropic pressures outside the galaxy plane, it is required to estimate the functional parameter α_{eff}

outside the galaxy plane. To achieve such estimate it must be defined each individual body contribution in the gravitational system to the metric shift function. Also aiming both at describing gravitational systems by more detailed models including the individual known massive bodies [17], as well as to achieve a physical interpretation for the functional parameter of the ELA metric, it is desirable to have a functional parameter for each individual body in any given gravitational system. Hence, next, we are mapping the shift function $N_{ELA,eff}^r$ into an equivalent product of the many body contributions defined with respect to the gravitational fields of each individual body in the gravitational system. To achieve such factorization let us note that the event horizons for each of the many body is no longer a spherical surface, instead the Schwarzschild surface for each of these n bodies is defined by the 2-dimensional solution to the equation $U_N = 1$ in the neighbourhood of each of the point-like massive bodies. Specifically, for a given body n , the event horizon is the solution of the following equation

$$\frac{2GM_n}{c^2|r_{1,n} - r_{1,SC,n}|} + \sum_{i \neq n} \frac{2GM_i}{c^2|r_{1,i} - r_{1,SC,n}|} = 1 . \quad (5)$$

The solution of the radial coordinate $r_{1,SC,n}$ for each of the N event horizons can be parameterized by the angular coordinates θ and φ such that, for each of the n bodies, this equation can be expressed in terms of a modified gravitational potential as $\tilde{U}_n = 1$. Specifically, this modified gravitational potential, is defined as

$$\begin{aligned} \tilde{U}_n &= \frac{2G\tilde{M}_n(\theta, \varphi)}{c^2|r_{1,n} - r_1|} , \\ \tilde{M}_n(\theta, \varphi) &= M_n + |r_{1,n} - r_{1,SC,n}(\theta, \varphi)| \sum_{i \neq n} \frac{M_i}{|r_{1,i} - r_{1,SC,n}(\theta, \varphi)|} . \end{aligned} \quad (6)$$

For gravitational systems for which the values of the Schwarzschild radius is much smaller than the distance between bodies, $|r_{1,n} - r_{1,SC,n}(\theta, \varphi)| \ll |r_{1,i} - r_{j,n}(\theta, \varphi)|$, this modified gravitational potential approximately matches the SC gravitational potential $\tilde{U}_n \approx U_n$.

Once we have defined these modified potentials, the shift function for the ELA metric can be factorized into a product of exponentials $(1 - \tilde{U}_n)$ which exactly vanish at the event horizon corresponding to each of the many body in the system and asymptotically converge to unity at spatial infinity, hence having the same properties of the original single body ELA metric. Specifically

we obtain the ansatz

$$\begin{aligned}
N_{ELA.N}^r &= -\frac{Hr}{c} \prod_{n=1}^N (1 - \tilde{U}_n)^{\frac{\alpha_n}{2} + \frac{1}{2}} \\
ds_{ELA.N}^2 &= (1 - U_N) (cdt)^2 - d\Omega^2 \\
&\quad - (dr_1 + N_{ELA.N}^r (cdt))^2,
\end{aligned} \tag{7}$$

The map between the functional parameters of metrics (7) and (4) is straight forwardly obtained to be

$$\alpha_{\text{eff}} = -1 + \frac{\sum_{n=1}^N (\alpha_n + 1) \ln(1 - \tilde{U}_n)}{\ln(1 - U_N)}. \tag{8}$$

With respect to the gravitational interactions corrections, these two parameters have a distinct meaning. While the functional parameter α_n describes the gravitational interactions due to each body n in the remaining $N - 1$ bodies, the functional parameter α_{eff} describes the local effect near each body n due to the gravitational interactions of the remaining $N - 1$ bodies.

Most astrophysical systems, such as galaxies, lie approximately on a 2-dimensional plane. Hence the physical laws derived from direct observational data of a given astrophysical system are valid within such plane. The analysis of physical laws in the neighbourhood of the system is usually inferred from known gravitational laws as well as by analysing indirect observational data such as the gravitational lens effect. Following this discussion, from the map (8), known a profile for the functional parameter α_{eff} it is possible to derive a profile for each of the functional parameters α_n aiming at obtaining a estimation for the gravitational corrections due to the ELA metric outside of the plane of the astrophysical system.

For a given gravitational system we are considering the following simplification assumptions:

- the system is approximately planar allowing for a circular symmetric planar model;
- each of the functional parameters α_n is circular symmetric with respect to the centre of mass $r_{1.n}$ of each body M_n in the plane of the system;

Given these assumptions the map (8) allows to obtain a scaling law for the functional parameter α_n with respect to each body mass M_n . Let us consider N_M bodies, each with a mass M_0 , hence with total mass $M = N_M M_0$. If we consider these N_M bodies to be infinitesimally close to each other at a distance $r_{1.0}$ from a given test mass such that $U_{i_N} = U_0 = 2GM_0/(c^2 r_{1.0})$ and $\tilde{U}_{i_N} = NU_0$ (for $i_N = 1, \dots, N$), the effective functional parameter α_{N_M} is,

according to map (8), $\alpha_{N_M} = -1 + N(\alpha_0 + 1)$. Further noting that $N_M = M/M_0$ and that the N_M bodies must be consistently described by one single body of mass M we obtain the scaling law for the functional parameters α_n with respect to some reference mass M_0 to be

$$\alpha_n = -1 + \frac{M_n}{M_0}(\alpha_0 + 1) . \quad (9)$$

In particular, assuming that this scaling law is circular symmetric, allows to obtain a reference profile for the functional parameter α_0 on the planar system from a known profile for α_{eff} , also on the planar system. Further assuming either approximately spherical symmetry for each functional parameter α_n or by parameterizing its anisotropy along the orthogonal direction to the planar system the functional parameter α_{eff} can be computed outside the planar system. For a given many body system, we will explore this relation to estimate a 3D map for α_{eff} . We also remark that this scaling law implies relatively large values of the parameter α_M for large masses as it scales linearly with the mass.

In addition, assuming the above scaling relation for each of the many bodies with masses M_n , in the limit of large radii the map (8) is approximately given by

$$\begin{aligned} \alpha_{\text{eff}}(r_1 \sim +\infty) &= -1 + \sum_{n=1}^N \frac{M_n}{M_0} \frac{\tilde{M}_n(\theta, \varphi)}{M_N} (\alpha_0(r_1 \sim +\infty) + 1) \\ &\approx -1 + \sum_{n=1}^N \frac{M_n^2}{M_0 M_N} (\alpha_0(r_1 \sim +\infty) + 1) , \quad (10) \\ M_N &= \sum_{n=1}^N M_n , \end{aligned}$$

where M_N is the total mass of the gravitational system, the masses $\tilde{M}_n(\theta, \varphi)$ are defined in (6) and the last expression for α_{eff} is obtained by considering the approximation $\tilde{M}_n \approx M_n$. Noting that far from the astrophysical body the gravitational potential approximately equals the gravitational potential of a single point-like massive body with the total mass of the N-body astrophysical system $U_N \approx 2G M_N / (c^2 r_1)$ such that the mass contribution due to the extended anisotropic gravitational background is expressed by equation (2) with $M = M_N$ and further recalling that, as discussed in the introduction, to ensure a finite mass contribution the metric functional parameter must either be asymptotically null at spatial infinity or strictly null above a finite radial cut-off, we conclude that the above asymptotic expression for $\alpha_{\text{eff}}(r_1 \sim +\infty)$ must be

null. Hence, allowing for a generic value for the reference mass M_0 , we obtain

$$\alpha_{\text{eff}}(r_1 \sim +\infty) = 0 \Leftrightarrow \begin{cases} \alpha_0(r_1 \sim +\infty) = -1 + M_0 M_N \left(\sum_{n=1}^N M_n^2 \right)^{-1}, \\ \alpha_n(r_1 \sim +\infty) = -1 + M_n M_N \left(\sum_{n=1}^N M_n^2 \right)^{-1}. \end{cases} \quad (11)$$

These limits constitute a consistency check for the map (8) and the definition of the reference profile α_0 , in particular allow to set the asymptotic limit of such profile for large radii.

We note that when considering a detailed model containing the known massive bodies within a given gravitational system the ansatz (7) describing each individual body contribution to the background is applicable requiring a full numerical simulation with N functional parameters. However for a simplified model considering a planar average mass surface density, the ansatz (4) describing one single effective functional parameter on the galaxy plane reduces significantly the complexity of the model allowing for a simpler parameterization of gravitational corrections. Hence in the remaining of this work we will employ the effective metric ansatz (4) to develop a simplified disk lattice model for galaxies similar to cosmological lattice models [24]. For a given galaxy, once a discretized $2D$ profile for the functional parameter α_{eff} is obtained by fitting the rotation curve on the plane of the galaxy, the discretized profile for the functional parameter of each body on the galaxy model, α_n (7), can be parameterized by the scaling law for α_M (9) such that solving the system of equations corresponding to map (8) we obtain the discretized profile for α_0 on the galaxy plane corresponding to the model considered. From this discretized profile it is straight forward to explicitly compute the discretized profile for each of the α_n , also on the galaxy plane, from the scaling law (9). Further assuming either spherical symmetry (or a parameterization of the anisotropy) of the parameter α_n with respect to the centre of mass of each body in the model the, value for the functional parameter α_{eff} outside of the galaxy plane is computed by employing the map (8), hence allowing for a $3D$ analysis of derived gravitational quantities in the neighbourhood of the galaxy such as mass-energy, anisotropic pressures and equation of state corrections due to the ELA metric background.

When computing the equations of motion for either the metric (4) or (7), there is one more problem to address, although for the SC metric we have considered the series expansion to first order in the gravitational field for the metric components $g_{SC.00} = 1 - U_N$ and $g_{SC.r\bar{r}} = 1$, we have not considered a series expansion for the shift function N_{ELA}^r . Although for small values of the metric functional parameter $|\alpha| \sim 10$ an expansion of the factor $(1 - U)^\alpha$ to the same order of the metric component g_{00} maintains the accuracy of the ELA metric approximation, for higher values of the functional parameter a higher order expansion of the shift function on the gravitational field U is required to attain the same accuracy. We recall that a similar problem is verified in the

PN formalism as the component g_{00} requires an higher order expansion than the remaining metric components [18]. Although an higher order expansion of the ELA metric shift function can be considered, there is no clear technical advantage as it will difficult the derivation of the equations of geodesic motion and other derived quantities. Instead, for technical simplification we will carry the following computations to first order on the gravitational field with respect to the local metric background components $g_{SC,00} = 1 - U_N$ and $g_{SC,rr} = 1$ while considering the exact expression for the shift function N_{ELA}^r . Also this construction can be justified by noting that the shift function is here a functional parameter, hence a generic function for which the specific dependence on the gravitational field is unknown such that the specific order of its series expansion in the gravitational field cannot be exactly determined. In addition we will consider a series expansion for the metric and remaining derived quantities of second order on the Hubble rate H , hence of second order on the shift function N_{ELA}^r .

Given this modelling setup we proceed to compute the metric connections and obtain the dominant contributions to the radial acceleration of a test mass on the many body background described by the ansatz (4). Considering that the many body system lies approximately on a plane of constant θ and for non-relativistic velocities $\dot{r}_1 \ll c$, $\dot{\varphi}_1 \ll c$ and $\dot{\theta}_1 \ll c$, the radial acceleration approximated to first order on the gravitational field U_N and second order on the Hubble rate H is

$$\begin{aligned}
\ddot{r}_1 &\approx -c^2 \Gamma_{tt}^r - \Gamma_{\varphi\varphi}^r \dot{\varphi}^2 \approx F_N + F_\varphi + F_{H^2} + O(U_N^2, H^4) , \\
F_N &= -\frac{c^2}{2} U_N' , \quad F_\varphi = r_1 \dot{\varphi}^2 \sin \theta \\
F_{H^2} &= -\frac{c^2}{2} (N_{ELA,\text{eff}}^r)^2 U_N' \\
&\quad + c^2 N_{ELA,\text{eff}}^r \dot{N}_{ELA,\text{eff}}^r + c \dot{N}_{ELA,\text{eff}}^r \\
&= \frac{H^2 r_1}{2} (1 - U_N)^{\alpha_{\text{eff}}} \left((1 - U_N) (2 + r_1 \log(1 - U_N) \alpha'_{\text{eff}}) \right. \\
&\quad \left. - 2(1 + q)(1 - U_N)^{\frac{1}{2} - \frac{\alpha_{\text{eff}}}{2}} - r_1 (2 + \alpha_{\text{eff}} - U_N) U_N' \right) ,
\end{aligned} \tag{12}$$

where dotted quantities represent differentiation with respect to the time coordinate t and primed quantities differentiation with respect to the radial coordinate r_1 . The acceleration component F_N is the standard gravitational Newton acceleration, F_φ is the standard centripetal acceleration and F_{H^2} is the lower order correction in the Hubble rate H due to the expanding locally anisotropic background as described by the ELA metric (4).

As long as the parameter α_{eff} is finite, for small values of r_1 , $F_{H^2} \approx 0$ is negligible, while for large enough values of r_1 the acceleration component F_{H^2} is positive being approximated by the usual RW acceleration $\lim_{r_1 \rightarrow \infty} F_{H^2} \approx$

$F_{RW} = -qH^2 r_1$, hence coinciding with the cosmological acceleration outwards the observer. In between these two asymptotic limits, depending on the value of the functional parameter α_{eff} , the component F_{H^2} may either be positive or negative. We note that, by numerical inspection of F_{H^2} , for large negative values of the parameter α_{eff} the correction to the Newton gravitational acceleration is negative, hence towards the central mass [16]. In particular the metric functional parameter allows for a covariant parameterization of the excess acceleration towards the core of galaxies, hence increasing the orbital speed of massive bodies on the galaxy and allowing to describe the experimentally observed flattening of galaxy rotation curves [4, 7]. In the next section we explicitly derive a discrete disk galaxy model approximately matching the physical parameters of the galaxy UGC2885.

As for the mass-energy density for the many body ELA metric (4) it is

$$\begin{aligned}
\rho_\alpha &= \frac{c^2}{8\pi G r_1^2} (1 - U_N) \left(N_{ELA.\text{eff}}^r (N_{ELA.\text{eff}}^r + 2r_1 \partial_r N_{ELA.\text{eff}}^r) \right. \\
&\quad \left. - \frac{1}{4} \left((\partial_\theta N_{ELA.\text{eff}}^r)^2 + \frac{1}{\sin^2 \theta} (\partial_\varphi N_{ELA.\text{eff}}^r)^2 \right) \right) \\
&= \frac{3H^2}{8\pi G} (1 + U_N)(1 - U_N)^{\alpha_{\text{eff}} - 1} \left((1 - U_N)^2 \times \right. \\
&\quad \times \left(1 + \frac{r_1}{3} \log(1 - U_N) \partial_r \alpha_{\text{eff}} \right) - \frac{1 - \alpha_{\text{eff}}}{48} \left((1 - \alpha_{\text{eff}})(\partial_\theta U_N)^2 \right. \\
&\quad \left. \left. + \frac{1 - \alpha_{\text{eff}}}{\sin^2 \theta} (\partial_\varphi U_N)^2 + 16r_1(1 - U_N) \partial_r U_N \right) \right) \quad (13)
\end{aligned}$$

In the limit of large radii, as long as α_{eff} is finite, this expression coincides with the RW mass-energy density $\rho_{RW} = 3H^2/(8\pi G)$.

3 A Lattice model for galaxies

In this section we will apply the many body ELA metric (4) to develop a simplified lattice disk model for galaxies showing that the flattening of galaxy rotation curves can be fully parameterized by this ansatz. To model the galaxy baryonic matter we are considering a discrete polar lattice on the galaxy plane with point-like masses at each lattice vertex. Although it is possible to implement an analytical model on the plane of the galaxy, to compute the values of the effective functional parameter α_{eff} (8) outside the galaxy plane employing the method described in the previous section it is required to consider a gravitational system of point like masses as it employs a reference profile α_0 for each massive body (9). We also note that an explicit discretization of the mass-energy density simplifies the computational simulation of the galaxy model as we will develop in the remaining of this section. To explicitly compute the values of

the point-like masses on galactic plane we consider the same model employed in [5] consisting of an infinitesimal $2D$ thin exponential disk [19] describing the intergalactic gas and a $3D$ thin exponential disk [22] describing stellar matter across the galaxy plane plus a bulge matter distribution near the centre of the galaxy describing the galaxy core. The galaxy disk is considered to be finite having an explicit cut-off describing the galaxy edge [23].

For exemplification purposes we are modelling the galaxy UGC2885. Experimental photometric data and red-shift measurements for this galaxy are available in the works [4, 7–10]. We note that within the analysis carried in these references there are discrepancies in the derived physical parameters of the galaxy. These discrepancies are mainly due to the distinct estimates for the Heliocentric distance D , which is derived from the systemic velocity V_{sys} which, in turn, is computed from the average of the measured red-shift. While in [4, 7] it is considered the uncorrected systemic velocity $V_{sys} = 5794 \text{ km s}^{-1}$ corresponding to a Heliocentric distance of $D = 118 \text{ Mpc}$, in [8, 9] the systemic velocity is converted to the motion relative to the local group and the cosmic microwave background $\bar{V}_{sys} = 5683 \text{ km s}^{-1}$ corresponding to the Heliocentric distance of $D = 76 \text{ Mpc}$. In addition, given a estimate for \bar{V}_{sys} , the estimate for D also depends on the today's Hubble rate value H_0 and deceleration parameter q_0 which has been updated [13] with respect to the values considered in these references. The mass modelling of the galaxy is sensitive to several of the physical parameters, hence we re-derive these parameters considering the converted systemic velocity $\bar{V}_{sys} = 5683 \text{ km s}^{-1}$ [8, 9] and today's value for the Hubble rate $H_0 = 2.28 \times 10^{-18} \text{ s}^{-1}$ and deceleration parameter $q_0 = -0.582$. The derived physical parameters for the galaxy UGC2885 are listed in table 1.

Following the modelling setup discussed in [5], to explicitly define the gas and stellar matter surface densities on the galaxy plane we are considering the neutral Hydrogen H_I 21 *cm*-line photometric measurements analysed in [7] and the K-band photometric measurement analysed in [8]. The gas matter surface density is modelled by a thin $2D$ exponential disk matching the H_I matter surface density multiplied by the corrective factor of 1.4 [5] and the stellar matter surface density is modelled by integrating along the orthogonal spatial direction to the galaxy plane the $3D$ exponential disk matching the K-band luminosity in [5]. As for the galaxy bulge we directly fit the excess mass required to the rotation curve up to 28 kpc , hence describing the rising of this curve near the galactic core. In addition, due to the distinct estimates for D in [7] and [8], it is required to scale the surface densities computed in these references accordingly.

parameter		value	ref
M_{tot}	total mass	$1.3 \times 10^{12} M_{\odot}$	[9]
M_{tot}/L_B	mass to light ratio	$4.7 M_{\odot}/L_{\odot}$	[4]
M_{H_I}/M_{tot}	fractional H_I mass	$\approx 2.3\%$	[7]
M_{DB}/M_{tot}	disk fractional stellar mass	$\approx 21\%$	[7, 9]
$V_{\text{rot.lim}}$	rotational velocity at R_{max}	$298 (km s^{-1})$	[4, 7]
V_{sys}	measured systemic velocity	$5794 (km s^{-1})$	[9]
\bar{V}_{sys}	converted systemic velocity	$5683 (km s^{-1})$	[9]
R_{max}	galaxy semi-major axis	$83.89 (kpc)$	[4, 7, 9]
z	cosmological red-shift	0.01896	[13]
D	Heliocentric distance	$81.14 Mpc$	[13]
H	Hubble rate	$2.26 \times 10^{-18} s^{-1}$	[13]
q	deceleration parameter	-0.584	[13]

Table 1: Physical parameters of the galaxy UGC2885. The systemic velocity \bar{V}_{sys} is converted to the heliocentric velocity with respect to the local group and the cosmic microwave background [9] and the remaining quantities R_{max} , z , D , H and q , are computed for the most recent estimates for H_0 and q_0 [13].

Hence we obtain

$$\begin{aligned}
\mu &= \mu_D + \mu_B + \sum_{i=0}^{N_{28}} \mu_S[i], \\
\mu_D &= 0.00791432 e^{-\left(\frac{r_1}{37.5662 \text{ kpc}}\right)^2} \delta_{\text{cut-off}}(r_1) \quad (kg m^{-2}), \\
\mu_B &= 0.329868 e^{+\frac{r_1}{11.6367 \text{ kpc}}} \delta_{\text{cut-off}}(r_1) \quad (kg m^{-2}), \\
\delta_{\text{cut-off}}(r_1) &= \begin{cases} 1 & , \quad r_1 \leq R_{\text{max}} \\ 1 - \frac{r_1 - R_{\text{max}}}{\delta} & , \quad R_{\text{max}} < r_1 < R_{\text{max}} + \delta \\ 0 & , \quad r_1 \geq R_{\text{max}} \end{cases}
\end{aligned} \tag{14}$$

where, for a given radial coordinate discretization $r_{1[i]}$, $\mu_S[i]$ are constant surface density disks with edge at $r_1 = (r_{1[i]} + r_{1[i+1]})/2$, N_{28} corresponds to the integer labelling of the discretized radial coordinate nearest to 28 kpc and the function $\delta_{\text{cut-off}}$ describes a smooth cut-off for the galaxy edge near R_{max} .

We now proceed to explicitly define a lattice model for the galaxy UGC2885 similar to models considered for many body cosmological simulations [24]. Aligning the z axis orthogonally to the galaxy plane such that the galaxy disk lays in the plane of constant $\theta = \pi/2$ and considering a regular polar lattice discretization over the plane of the galaxy by i_{max} points along the radial coordinate r_1 and k_{max} points along the angular coordinate φ we obtain the discrete polar

coordinates for the lattice points labelled by $[i, k]$

$$\begin{aligned} r_{1[i]} &= i \times \Delta_{r_1} , \quad i = 0, 1, \dots, i_{max} , \\ \varphi_{[k]} &= \begin{cases} 2\pi & , \quad i = 0 \\ i \neq 0 , \quad k \times \Delta_k & , \quad k = 0, 1, \dots, k_{max} - 1 \quad , \quad \Delta_k = \frac{2\pi}{k_{max}} . \end{cases} \end{aligned} \quad (15)$$

The lattice faces are centered at each lattice point $[i, k]$, hence being defined as bounded $2D$ surfaces, and at each lattice point $[i, k]$ we consider a point-like mass $M_{[i]}$ of value matching the integrated surface mass-energy density over the lattice face $_{[i,k]}$

$$\begin{aligned} \text{face}_{[i,j]} &= \left[r_{1[i]} - \frac{\Delta_{r_1}}{2}, r_{1[i]} + \frac{\Delta_{r_1}}{2} \right] \times \left[\varphi_{1[k]} - \frac{\Delta_k}{2}, \varphi_{1[k]} + \frac{\Delta_k}{2} \right] , \\ M_{[0]} &= 2\pi \int_0^{\frac{\Delta_{r_1}}{2}} dr_1 r_1 \mu(r_1) , \\ M_{[i]} &= \Delta_k \int_{r_{1[i]} - \frac{\Delta_{r_1}}{2}}^{r_{1[i]} + \frac{\Delta_{r_1}}{2}} dr_1 r_1 \mu(r_1) , \end{aligned} \quad (16)$$

where μ is the modelled surface mass density (14). Hence we are considering $N = 1 + i_{max} \times k_{max}$ point-like massive bodies over the planar polar lattice.

Assuming that all the bodies are approximately at a stable circular orbit the galaxy model exhibits explicit planar circular symmetry on the plane of galaxy such that both the orbital velocities and the gravitational potentials for the bodies at each lattice point $[i, k]$ are independent of the angular coordinate index k . Hence, noting that the time derivative of the angular coordinate φ for each value of the radial coordinate $r_{1[i]}$ is $\dot{\varphi}_{\text{orb}[i]} = V_{\text{orb}[i]}/r_{1[i]}$, the radial equations of motion (12) on the galaxy plane are

$$\ddot{r}_{1[i]} = 0 \Leftrightarrow V_{\text{orb}[i]} = \sqrt{-r_{1[i]}(F_{N[i]} + F_{H^2[i]})} , \quad (17)$$

and the the potential U and its derivative U' have the following explicit expressions at each lattice point

$$\begin{aligned} U_{[i]} &= \sum_{[j,k] \neq [i,0]} \frac{2GM_{[i]}}{c^2 \Delta r_{1[i,0][j,k]}}, \\ U'_{[i]} &= - \sum_{[j,k] \neq [i,0]} \frac{2GM_{[i]}(r_{1[i]} - r_{1[j]} \cos(\varphi_{[k]}))}{c^2 (\Delta r_{1[i,0][j,k]})^3} , \\ \Delta r_{1[i,0][j,k]} &= \sqrt{r_{1[i]}^2 - 2r_{1[i]}r_{1[j]} \cos(\varphi_{[k]}) + r_{1[j]}^2} . \end{aligned} \quad (18)$$

In these expressions we are considering as the test masses the bodies at the lattice points $[i, 0]$. Due to the circular symmetry of the planar model any other

value of k for the test masses may generally be considered. As for the functional parameter α_{eff} on the galactic plane we consider it to be a radial symmetric function such that its values is also independent of the angular lattice index k . In the following we will fit this parameter to the experimentally observed galaxy rotation curves, hence it is further required to consider a numerical derivative to evaluate the derivative of this parameter with respect to the radial coordinate α'_{eff} . We consider a first order approximation to this derivative. To reduce the numerical uncertainty on this approximation we further consider the parameterization of the functional parameter by a second order expansion on the gravitational field, $\alpha_{\text{eff}} = 3 + \alpha(1 - U)^2$. This parameterization does not necessarily has any physical meaning, it is employed here as a parameterization that effectively reduces the relative magnitude of the contribution of the first order discrete derivative, hence allowing to numerically fit the functional parameter of the model to the existing velocity profiles without requiring to consider an higher order approximation to the derivative. Therefore, at each lattice point $[i, k]$ on the plane of the galaxy (again independently of the index k) we obtain the following expressions for the functional parameter and its derivative with respect to the radial coordinate

$$\begin{aligned}\alpha_{\text{eff}[i]} &= 3 + \alpha_{[i]}(1 - U_{[i]})^2, \\ \alpha'_{\text{eff}[i]} &= -2U'_{[i]}(1 - U_{[i]}) + (1 - U_{[i]})^2 \frac{\alpha_{[i+1]} - \alpha_{[i]}}{r_{1[i+1]} - r_{1[i]}},\end{aligned}\tag{19}$$

where we have considered forward finite differences when writing the discrete derivative of the functional coefficient $\alpha_{[i]}$. This construction allows to describe the unknown functional parameter that parameterizes the gravitational interactions corrections by its values across the discretized lattice points. As for the gravitational acceleration contributions to the orbital velocity (17) are

$$\begin{aligned}F_{N[i]} &= -\frac{c^2}{2}U'_{[i]}, \\ F_{H^2[i]} &= \frac{H^2 r_{1[i]}}{2} (1 - U_{[i]})^{\alpha_{[i]}} \left(2(1 - U_{[i]}) \right. \\ &\quad \left. - 2(1 + q)(1 - U_{[i]})^{\frac{1}{2} - \frac{\alpha_{[i]}}{2}} \right. \\ &\quad \left. - r_{1[i]}(2 + \alpha_{[i]} - U_{[i]})U'_{[i]} \right) + \Delta F_{H^2[i]} \\ \Delta F_{H^2[i]} &= \frac{(H r_{1.[i]})^2}{2} (1 - U_{[i]})^{\alpha_{\text{eff}[i]} + 1} \log(1 - U_{[i]}) \alpha'_{\text{eff}[i]}.\end{aligned}\tag{20}$$

$V_{\text{orb}[i]}$ 17 can now be fitted to the experimental measured rotational velocities. At each lattice point the values for the experimentally measured velocity $V_{\text{exp}[i]}$ are linearly interpolated for each value of the radial coordinate $r_{1[i]}$ to the experimental profile listed in table 3 of reference [7]. To numerically evaluate the

values of the functional parameter α_{eff} that fit this data we are considering an edge smoothing of $\delta = 5 \text{ kpc}$ and a discretization of the radial coordinate on the galaxy plane up to $R_{\text{max}} + \delta = 88.89 \text{ kpc}$ with the following lattice properties

$$\begin{aligned}
i_{\text{max}} &= 74, \\
k_{\text{max}} &= 16, \\
\Delta_{r_1} &= 1.20464 (\text{kpc}), \\
\Delta_k &= \frac{\pi}{8}, \\
N &= 1185.
\end{aligned} \tag{21}$$

For values of $\alpha_{\text{eff}[i]} \sim -10^7$ on the galaxy disk the flattening of the galaxy rotation curve is fully accounted for. The velocity contributions due to the Newton gravitational acceleration $V_{N[i]} = \sqrt{-r_{1[i]} F_{N[i]}}$ and due to the background described by the ELA metric $V_{\alpha[i]} = \sqrt{-r_{1[i]} (F_{H^2[i]} + \Delta F_{H^2[i]})}$ to the total orbital velocity $V_{\text{orb}[i]} = \sqrt{V_{N[i]}^2 + V_{\alpha[i]}^2}$ are plotted in figure 1. The values for these velocities as well as the values of the point-like baryonic masses M_i and the coefficients $\alpha_{[i]}$ are listed in table 2 in the appendix. We note that reducing

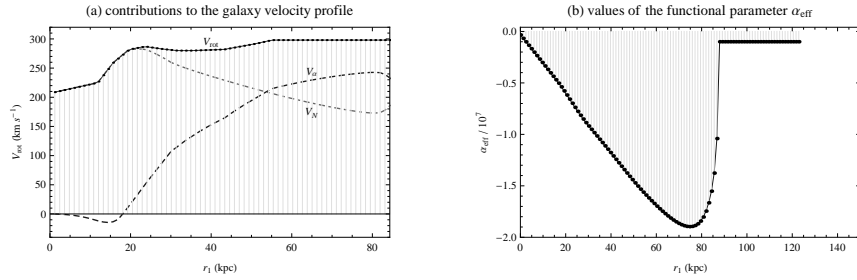


Figure 1: **(a)** Velocity profiles for UGC2885. The velocity contribution due to the Newton acceleration (V_N) is represented by a dashed-dot line, the velocity contribution due to ELA metric correction to the gravitational acceleration (V_α) is represented by a dashed line and the total orbital velocity (V_{orb}) by a continuous line with the lattice points represented by dots; **(b)** profile for the functional parameter α_{eff} on the galactic plane, the lattice points are represented by dots.

the spacing of the lattice, hence increasing the number of the bodies in the lattice model and reducing the mass for each of these massive bodies, does not change significantly the value of the effective functional parameter α_{eff} . This is mainly due to the shift function depending on the total gravitational potential U_N which is approximately maintained constant for each value of the

radial coordinate independently of how many discretization points have been considered.

The total baryonic mass for this model is

$$M_b = 2.30 \times 10^{11} M_\odot . \quad (22)$$

However there are contributions to the total galaxy matter due to mass-energy density contribution due to the expanding locally anisotropic background corrections with respect to Schwarzschild backgrounds [16]. In particular it is relevant to remark that to ensure that causality is preserved the total mass-energy must be strictly positive both on the galaxy disk as well as beyond the galaxy edge. For the particular fit discussed here we have considered that beyond the galaxy disk edge the value of the parameter α_{eff} is approximately a constant $\sim 10^{-6}$ ensuring that the mass-energy density of the background is strictly positive. Next we analyse and discuss in detail the mass-energy densities for the galaxy model.

4 Mass-energy density analysis

To evaluate the mass-energy density ρ_α due to the locally anisotropic background within the galaxy plane it is enough to evaluate the expression (13). We have already fitted the functional parameter α_{eff} to the rotation curve on the plane of the galaxy as pictured in figure 1, hence it is straight forward to evaluate $\rho_{\alpha[i,k]}$ at each lattice point $[i,k]$ by excluding the point-like mass at this lattice point and consider the contribution of the remaining $N - 1$ point-like masses. Noting that the derivatives of the gravitational potential $\partial_\theta U$ and $\partial_\varphi U$ are null and that ρ_α is circular symmetric on the galaxy plane, the discrete quantities $U_{[i]}$, $U'_{[i]}$ (18), α_{eff} and α'_{eff} (19) already computed are enough to actually compute the profile for the mass-energy density $\rho_{\alpha[i,0]}$ on the galaxy plane. Such profile is plotted in figure 2.

We can readily verify that there is a negative mass-energy contribution due to the expanding locally anisotropic background for radial distances greater than 21 kpc from the centre of the galaxy. Strictly negative mass-energy densities violate causality, therefore are commonly not considered as a physical reality. For the specific model discussed here the mass-energy is strictly positive on the galaxy plane up to the distance of 89 kpc from the galaxy centre such that causality is preserved. To show it explicitly let us note that the surface mass-energy contributions μ_B and μ_D (14) are effective 2D quantities describing the 3D mass-energy densities ρ_B and ρ_D , respectively. Specifically ρ_B is fitted to a exponential 3D disk in [8] and for ρ_D we assume that it is approximately described by a Gaussian disk of thickness $\Delta_{r_1}/2$

$$\begin{aligned} \rho_B &= 4.60183 \times 10^{-21} e^{-\frac{r_1}{11.6367 \text{ kpc}} - \frac{z}{1.16367 \text{ kpc}}} , \\ \rho_D &= 2.40691 \times 10^{-22} e^{-\frac{r_1}{37.5662 \text{ kpc}} - \left(\frac{z}{0.602322 \text{ kpc}}\right)^2} . \end{aligned} \quad (23)$$

Hence within the galaxy disk the sum of these two contributions plus the contribution ρ_α (13) is strictly positive. However, beyond the galaxy disk edge

$r_1 \geq 89 \text{ kpc}$, the baryonic mass-energy is approximately null such that the only contribution to the total mass-energy density is due to ρ_α . For the galaxy model presented here $\rho_\alpha > 0$ for $r_1 > 89 \text{ kpc}$ when $\alpha_{\text{eff}} \geq -10^6$. Also we recall that ρ_α does contribute to the total mass of the galaxy in addition to the baryonic mass such that to ensure a finite contribution mass for the model we further consider a upper radial cut-off R_α above which the functional parameter is null $\alpha_{\text{eff}} = 0$ such that explicit spatial isotropy is recovered and the mass-energy density above this cut-off exactly matches the one for the expanding Universe ρ_{RW} [11]. Hence both to preserve causality beyond the galaxy edge and to ensure a total finite galaxy mass we consider that the functional parameter α_{eff} is approximately a constant in-between the radial distance of 89 kpc and R_α

$$\alpha_{\text{eff}} = \begin{cases} -10^6 & , \quad r_1 \in]88.89 \text{ kpc}, R_\alpha] \text{ ,} \\ 0 & , \quad r_1 \in]R_\alpha, +\infty[\text{ .} \end{cases} \quad (24)$$

This ansatz is based only on the assumption that causality must be maintained as well as finitude of the total galaxy mass. A more realistic setup and estimate for α_{eff} would require the analysis of direct observational data outside the galaxy disk, for instance the gravitational lens effect on the background radiation due to the deformation of the gravitational background induced by the galaxy. We leave such analysis to another work, here we will proceed our analysis taking the ansatz (24) as an approximately constant lower cut-off for the functional parameter α_{eff} .

Given the model setup just described, the total mass-energy density is strictly positive on the galaxy plane both within the galaxy disk and beyond the galaxy edge. The contributions to the total mass-energy density on the galaxy plane are plotted in figure 2.

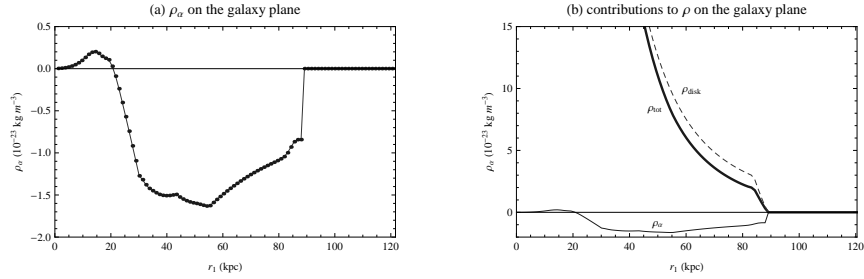


Figure 2: Mass-Energy profiles for the galaxy model: **(a)** mass-energy density ρ_α (13) on the galactic plane, the lattice point are represented by dots; **(b)** contributions ρ_α (thin line) and ρ_μ (dashed line) to the total mass-energy density ρ_{tot} (thick line) on the galaxy plane. The total mass-energy is strictly positive.

In addition to the analysis of the mass-energy contributions and the ansatz for the value of α_{eff} beyond the galaxy edge one is led to the question whether

our model can further be extended in a 3D neighbourhood of the galaxy, in particular whether it is possible to estimate the value of the functional parameter and the mass-energy density along the orthogonal direction to the plan of the galaxy. To implement such analysis let us consider the map between metric (4) and metric (7) and assume the existence of an unique reference profile α_0 corresponding to a reference mass of M_0 such that each for the functional parameters α_n are defined with respect to α_0 accordingly to the scaling law (9). If the 3D profile for α_0 is known also the 3D profiles for the several α_n 's can be computed from the scaling law (9) and the value for the parameter α_{eff} along the orthogonal direction to the galaxy plan can be estimated from the map (8) by considering some given symmetry for the several functional parameters α_n .

Noting that the galaxy model is explicitly circular symmetric, a discrete planar profile $\alpha_{0[i]}$ on the galaxy plane for the functional parameter α_0 can be computed by solving the linear system

$$\alpha_{\text{eff}[i]} = -1 + \frac{\sum_{[j,k] \neq [i,0]} \frac{M_{[j]}}{M_0} (\tilde{\alpha}_{0[i,0][j,k]} + 1) \log(1 - U_{n[i,0][j,k]})}{\log(1 - U_{[i]})}, \quad (25)$$

where the potentials $U_{n[i,0][j,k]}$ correspond to the gravitational potential of the point-like massive body located at the lattice point $[j, k]$ evaluated at the lattice point $[i, 0]$ and $\tilde{\alpha}_{0[i,0][j,k]}$ is the linear interpolated profile for $\alpha_{0[i]}$ evaluated at a distance $\Delta r_{1[i,0][j,k]}$ from $\alpha_{0[0]}$

$$\begin{aligned} U_{n[i,0][j,k]} &= \frac{2GM_{[j]}}{c^2 \Delta r_{1[i,0][j,k]}}, \\ \tilde{\alpha}_{0[i,0][j,k]} &= \begin{cases} \alpha_{0[\tilde{j}]} + \frac{(\Delta r_{1[i,0][j,k]} - r_{1[\tilde{j}]}) (\alpha_{0[\tilde{j}]} - \alpha_{0[\tilde{j}+1]})}{\Delta r_1}, & \text{for } \Delta r_{1[i,0][j,k]} \geq r_{1[\tilde{j}]} \\ \alpha_{0[\tilde{j}-1]} + \frac{(\Delta r_{1[i,0][j,k]} - r_{1[\tilde{j}-1]}) (\alpha_{0[\tilde{j}-1]} - \alpha_{0[\tilde{j}]})}{\Delta r_1}, & \text{for } \Delta r_{1[i,0][j,k]} < r_{1[\tilde{j}]} \end{cases}, \\ r_{1[\tilde{j}]} &= \text{nearest}(\Delta r_{1[i,0][j,k]}). \end{aligned} \quad (26)$$

The potentials $U_{[i]}$ and the distances $\Delta r_{1[i,0][j,k]}$ are defined in (18) and \tilde{j} is the index of the discrete radial coordinate nearest to $\Delta r_{1[i,0][j,k]}$.

To ensure that the system of equations (25) is not degenerate it is required that both the profile $\alpha_{\text{eff}[i]}$ and $\alpha_{0[i]}$ have the same length such that the indice i for both parameters run from 0 to some I_{MAX} . Here we are considering that $I_{MAX} = 5i_{max}$ with $\alpha_{\text{eff}[i > i_{max}]} = -10^6$. Hence, solving this linear system we obtain the profile $\alpha_{0[i]}$ on the galaxy plane. Such profile for our lattice disk galaxy model is plotted in figure 3.

Known the discrete reference profile for α_0 and some symmetry along the orthogonal direction to the galaxy plane it is straight forward to compute the

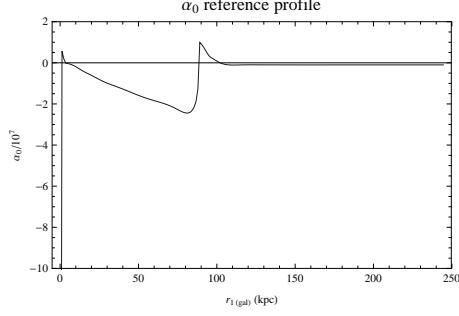


Figure 3: Discrete circular symmetric reference profile α_0 on the galaxy plane.

value of the functional parameter α_{eff} outside the galaxy plan. Specifically at a spatial point orthogonal to the lattice position $[i, 0]$ with a angle θ with respect to the z axis, hence corresponding to the spherical coordinates $(r_{1[i]}/\sin(\theta), 0, \theta)$, we obtain

$$\bar{\alpha}_{\text{eff}[i]}(\theta) = -1 + \frac{\sum_{[j,k]} \frac{M_{[j]}}{M_0} (\bar{\alpha}_{0[i]}(\theta) + 1) \log(1 - \bar{U}_{n[i,0][j,k]}(\theta))}{\log(1 - \bar{U}_{N[i]}(\theta))} , \quad (27)$$

where $\bar{U}_{n[i,0][j,k]}(\theta)$ is the gravitational potential of the massive body at the lattice position $[j, k]$ and $\bar{U}_{N[i]}(\theta)$ is the total gravitational potential of the N bodies, both evaluated at the position $(r_{1[i]}/\sin(\theta), 0, \theta)$

$$\begin{aligned} \bar{U}_{n[i,0][j,k]}(\theta) &= \frac{2GM_{[j]}}{c^2 \Delta \bar{r}_{1[i,0][j,k]}(\theta)} , \\ \bar{U}_{N[i]}(\theta) &= \sum_{[j,k]} \bar{U}_{n[i,0][j,k]}(\theta) , \\ \Delta \bar{r}_{1[i,0][j,k]}(\theta) &= \sqrt{\left(\frac{r_{1[i]}}{\sin \theta}\right)^2 - 2r_{1[i]}r_{1[j]} \cos(\varphi_{[k]}) + r_{1[j]}^2} . \end{aligned} \quad (28)$$

In this expressions $\bar{r}_{1[i,0][j,k]}(\theta)$ is the Euclidean distance between the lattice point $[j, k]$ (i.e. $(r_{1[j]}, \varphi_{[k]}, \theta)$) and the spatial position $(r_{1[i]}/\sin(\theta), 0, \theta)$.

As for $\bar{\alpha}_{0[i]}(\theta)$ it is the estimate for the value of the reference parameter α_0 evaluated outside the galaxy plane at the position $(r_{1[i]}/\sin(\theta), 0, \theta)$. To actually estimate the value of this parameter at a generic $3D$ position it is required to assume either spherical symmetry of each functional parameter α_n with respect to the centre of mass of body n or assuming an explicit anisotropy orthogonally to the galaxy plane proportional to $(\sin \theta)^{\xi_\theta}$ such that circular symmetry (for a fixed value of the coordinate θ) is explicitly maintained. Such anisotropy is consistent with the planar model for the galaxy as the matter density distribution (a planar disk) explicitly violates spherical symmetry maintaining circular

symmetry and it allows for a fine-tune of the mass-energy density maintaining it strictly positive outside the galaxy plane. In addition we note that as already discussed in the introduction, for large radial distances from the galaxy, the many body metric must consistently be asymptotically described by one single massive body with mass matching the total baryonic mass of the galaxy such that spherical symmetry is approximately recovered and the functional parameter α_{eff} is asymptotically independent of both the angular coordinates φ and θ . Hence we are considering the following assumptions:

- approximately circular symmetry such that all quantities are independent of the angular coordinate φ ;
- anisotropic dependence of the parameter α_0 on the factor $(\sin \theta)^{\xi_\theta}$ which, for the particular case of $\xi_\theta = 0$, corresponds to spherical symmetry;
- for large radial distances the parameter α_0 is independent of the angular coordinate θ approximately matching the value $\alpha_{0[I_{MAX}]}$.

Given these assumptions it is straight forward to define the parameter α_0 evaluated outside the galaxy plane at the position $(r_{1[i]}/\sin(\theta), 0, \theta)$

$$\bar{\alpha}_{0[i]}(\theta) = \alpha_{0[I_{MAX}]} + (\tilde{\alpha}_{0[i,0][j,k]}(\theta) - \alpha_{0[I_{MAX}]}) \left(\frac{\Delta \bar{r}_{1[i,0][j,k]}(\pi/2)}{\Delta \bar{r}_{1[i,0][j,k]}(\theta)} \right)^{\xi_\theta}, \quad (29)$$

where the sine of the angle $\theta_{[i,0][j,k]}(\theta)$ between the orthogonal direction to the lattice point $[j, k]$ and the line between this lattice point and the position $(r_{1[i]}/\sin(\theta), 0, \theta)$ is given by the ratio of the radial distances $\Delta \bar{r}_{1[i,0][j,k]}(\pi/2)/\Delta \bar{r}_{1[i,0][j,k]}(\theta)$ and $\tilde{\alpha}_{0[i,0][j,k]}(\theta)$ is the value of the linearly interpolated reference parameter α_0 corresponding to the distance to the origin of $\Delta \bar{r}_{1[i,0][j,k]}(\theta)$, i.e. the distance to $\alpha_{0[0]}$ (26).

For $\xi_\theta = 0$, corresponding to spherical symmetric α_n 's there exist regions outside the galaxy plane where the mass-energy density is negative, while for large values of this exponent, $\xi_\theta > 50000$, corresponding to anisotropic α_n 's, the mass-energy density is strictly positive everywhere. The map for both the functional parameter α_{eff} and the mass-energy density contribution ρ_α on a plane orthogonal to the galaxy containing the galaxy centre is plotted in figure 4. The several samplings of α_{eff} and ρ_α orthogonal to the galaxy plane up to the distance (on the galaxy plane) to the centre of the galaxy of $r_{1(gal)} = r_{1[61]} = 73.48 \text{ kpc}$ considered when building the map of figure 4 are plotted in figures 5 to 8 in the appendix.

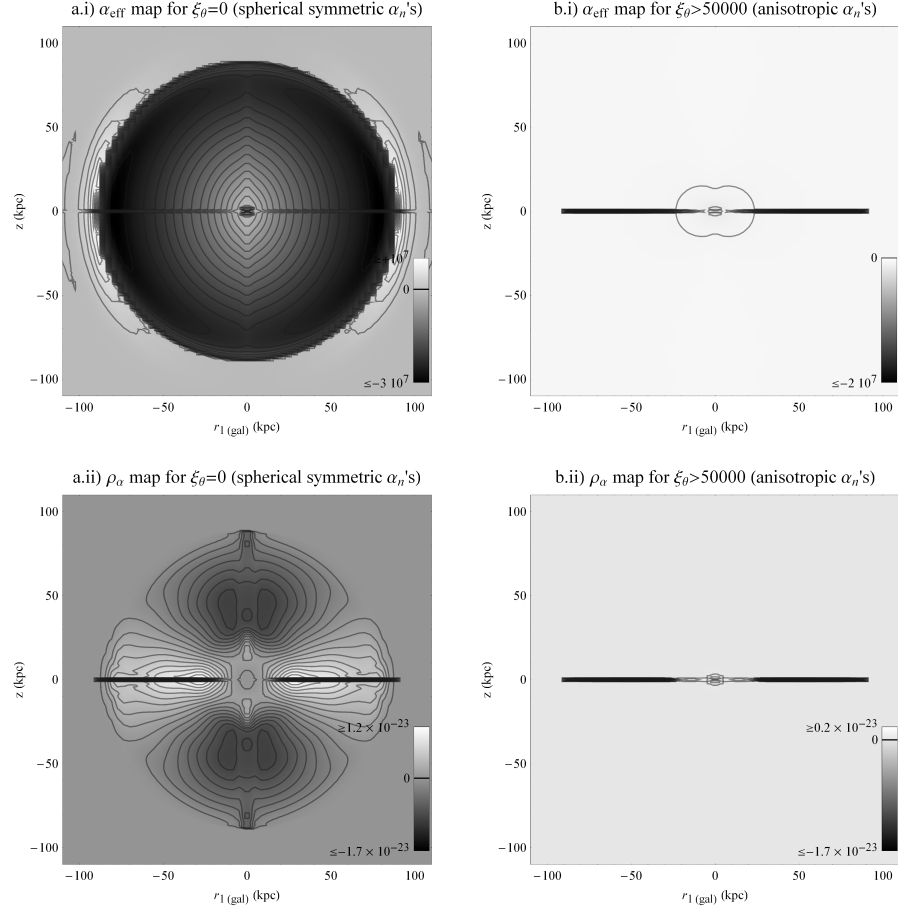


Figure 4: The map of the functional parameter α_{eff} and the mass-energy density ρ_α on a plane orthogonal to the galaxy containing the center of the galaxy, the x axis corresponds to the galactic plane such that $r_{1(\text{gal})}$ is the distance on the galaxy plane to the centre of the galaxy and the y axis corresponds to the orthogonal distance to the plane of the galaxy: **a)** assuming spherical symmetry, $\xi_\theta = 0$, for all α_n 's there exist regions outside the galaxy plane where the mass-energy is negative; **b)** assuming anisotropy along the orthogonal direction to the galaxy plane, for $\xi_\theta > 50000$, the mass-energy density is strictly positive outside the galaxy plane.

Noting that, independently of the direction, for large radial distances to the centre of the galaxy the functional parameter is approximately constant $\alpha_{\text{eff}} = -10^6$ up to the upper cut-off R_α (24), it is straight forward to estimate the relative mass-energy contribution for the relative cosmological mass-energy density corresponding to the galaxy model developed in this work. Let us consider a relative large distance from the galaxy such that the gravitational field of the galaxy is approximately given by one single point-like massive body, hence the mass contribution in addition to the expanding Universe background mass-energy density ρ_{RW} is

$$M_\alpha = 4\pi \int_0^{+\infty} r_1^2 (\rho_\alpha - \rho_{RW}) dr_1 = \frac{H^2}{2G} \left((1 - U(R_\alpha))^{\alpha_{\text{eff}}(R_\alpha)} - 1 \right) R_\alpha^3. \quad (30)$$

Assuming that all the baryonic matter has a similar contribution to the relative cosmological mass-energy contribution $M_\alpha \approx 4.96 M_b$ [13] we obtain for our galaxy model (22)

$$M_{\text{tot}} = 13.68 \times 10^{11} M_\odot, \quad M_\alpha = 11.38 \times 10^{11} M_\odot \Leftrightarrow R_\alpha = 9574.61 \text{ kpc} \quad (31)$$

Recalling that within the Λ CDM cosmological model such contribution must be included in the relative mass-energy density of dark matter, the extended background described by the ELA metric up to R_α is interpreted as the heuristic dark matter halo commonly employed to describe dark matter effects within galaxies.

5 Discussion and Outlook

By fitting the expanding locally anisotropic metric functional parameter α_{eff} (4) we have fully describe the flattening of the rotation curve for the galaxy UGC2885. Based on a lattice model we have estimate values of this functional parameter of order $\sim -10^{-7}$ within the galaxy disk simultaneously maintaining compatibility with the well established short-scale gravitational laws (the Schwarzschild metric is approximately recovered at planetary scales) and cosmological scale gravitational laws (the Universe expansion). Phenomenologically this is interpreted as a unaccounted gravitational interaction acting on baryonic matter that reproduces the effects attributed to cold dark matter, namely the flattening of galaxies rotation curves.

Also we have analysed the mass-energy density on the galaxy plane showing that although there are negative contributions from the extended background described by the ELA metric this quantity is strictly positive within the galaxy disk. Outside the galaxy disk, both on the plane of the galaxy and outside the galaxy plane, to ensure that the mass-energy density is strictly positive, it is considered a lower absolute value for the functional parameter (specifically $\sim -10^{-6}$). This construction implies that the metric functional parameter is larger in absolute value in the vicinity of baryonic matter, i.e. on the galaxy plane, than in the vicinity of the galaxy such that the gravitational corrections are

localized in the galaxy disk. In addition the functional parameter is null above the upper cut-off $R_\alpha = 9574.61 \text{ kpc}$ such that the ratio of the galaxy baryonic matter M_b to M_α , the mass contribution of the extended background in addition to the cosmological background matches the ratio of the baryonic to cold dark matter relative cosmological mass-energy densities $\Omega_b/\Omega_c = M_b/M_\alpha = 4.96$ consistently with the physical interpretation that the ELA metric constitutes a parameterization of cold dark matter.

A relevant question raised by the construction developed here is whether the gravitational corrections due to the expanding locally anisotropic background to the redshift of radiation emitted from the galaxy are significant or not [16]. To answer this question let us recall that when computing the profile for the functional parameter we have excluded the point-like masses at each lattice point as these are considered test masses on the background of the remaining $N - 1$ massive bodies on the galaxy. When considering an external point mass in the vicinity of each body on the galaxy, the effective functional parameter for the N body system, as perceived by the external test mass, must decrease in absolute value as near massive objects the Schwarzschild metric is asymptotically recover. Hence for a external test mass in the galaxy plane, near each massive body the functional parameter decreases in absolute value such that the red-shift corrections are negligible for radiation emitted from each body on the galaxy. As an analogy let us note that for many-body gravitational systems on Schwarzschild geometries, when computing orbits a given planet is considered a test mass such that its own gravitational field is not accounted in the equations of motion, however the frequency-shift for radiation emitted from each planet is mainly accounted for by the planet gravitational field.

As future directions of research we note that to develop a more detailed model for galaxies parameterized by a many body expanding locally anisotropic background it must be derived a continuous description of the intergalactic matter (gas) not contained on stars. Also, within a more detailed model, the assumption that all masses have a similar profile for the functional parameter given by the scaling law (9) may be an over simplification as the localization of the gravitational corrections in the vicinity of galactic plane indicates that the functional parameter for each of the many bodies on the galaxy may depend on the remaining masses in the gravitational system, hence not being exactly circular symmetric as considered in the simplified model developed in this work.

As a final remark let us note that a direct analysis in the neighbourhood of galaxies which may either confirm or dismiss the construction suggested here is achievable, for instance, by analysing the lens effect of the background radiation near galaxies. Also within the Solar System a test mass orbiting orthogonally to the planetary system plane would allow for a direct measurement of the localization of the gravitational corrections encoded in the expanding locally anisotropic metric. We leave these analysis to another work.

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A Appendix

In this appendix we list the values for the galaxy model masses, functional parameter and velocity contributions corresponding to figure 1 in table 2 and the samplings along the orthogonal to the galaxy plane of the functional parameter and mass-energy densities considered to plot the maps of figure 4 in figures 5 to 8.

i	$r_{1[i]}$ kpc	$M_{[i]}$ $10^8 M_{\odot}$	$\alpha_{\text{eff}[i]}$ 10^7	$V_N (km s^{-1})$	$V_{\alpha} (km s^{-1})$	$V_{\text{orb}} (km s^{-1})$
0	0	75.3802	0.0000003	0	0	0
1	1.20464	4.48560	-0.0335008	208.788	-0.143264	208.788
2	2.40929	4.73906	-0.067002	210.623	-0.434526	210.623
3	3.61393	4.78543	-0.100503	212.459	-0.911588	212.457
4	4.81858	5.05407	-0.134004	214.294	-1.612610	214.288
5	6.02322	4.99917	-0.167506	216.129	-2.584880	216.114
6	7.22787	5.22984	-0.201007	217.965	-3.848380	217.931
7	8.43251	5.15292	-0.234508	219.804	-5.457760	219.736
8	9.63715	5.38540	-0.268009	221.648	-7.377250	221.525
9	10.8418	5.32973	-0.301511	223.492	-9.671340	223.283
10	12.0464	5.63228	-0.335012	226.963	-12.06130	226.642
11	13.2511	5.64015	-0.368513	237.720	-13.82750	237.318
12	14.4557	5.58059	-0.402014	248.478	-14.62600	248.047
13	15.6604	5.36167	-0.435516	259.235	-13.40730	258.889
14	16.8650	5.05500	-0.469017	266.086	-9.44566	265.918
15	18.0697	4.81708	-0.502518	273.043	0.00000	273.043
16	19.2743	4.41684	-0.538329	279.600	10.7279	279.805
17	20.4790	4.04947	-0.577473	282.205	21.7037	283.038
18	21.6836	3.67934	-0.617500	282.686	32.7323	284.575
19	22.8882	3.37378	-0.657534	282.727	43.8787	286.112
20	24.0929	3.02316	-0.697586	281.168	54.9193	286.481
21	25.2975	2.76502	-0.737376	277.601	65.6203	285.252
22	26.5022	2.50679	-0.776331	273.602	76.2271	284.022
23	27.7068	2.31885	-0.814293	269.214	86.7563	282.847
24	28.9115	2.12502	-0.850928	264.624	97.2881	281.941
25	30.1161	1.99283	-0.885900	259.543	107.787	281.035
26	31.3207	1.87052	-0.918503	256.006	113.723	280.129
27	32.5254	1.75714	-0.950966	252.866	120.244	280.000
28	33.7300	1.65179	-0.983126	249.919	126.257	280.000
29	34.9347	1.55366	-1.01515	247.094	131.699	280.000
30	36.1393	1.46206	-1.04723	244.360	137.085	280.186
31	37.3440	1.37638	-1.07935	241.697	142.334	280.493
32	38.5486	1.29612	-1.11154	239.091	147.256	280.800
33	39.7533	1.22087	-1.14388	236.533	151.901	281.108
34	40.9579	1.15027	-1.17643	234.013	156.309	281.415
35	42.1625	1.08400	-1.20928	231.526	160.510	281.723
36	43.3672	1.02181	-1.24248	229.067	164.732	282.150
37	44.5718	0.963453	-1.27602	226.635	170.630	283.686
38	45.7765	0.908717	-1.30932	224.227	176.280	285.223
39	46.9811	0.857398	-1.34237	221.843	181.706	286.760
40	48.1858	0.809309	-1.37511	219.484	186.893	288.275
41	49.3904	0.764268	-1.40752	217.151	191.888	289.785
42	50.5951	0.722103	-1.43954	214.845	196.710	291.295
43	51.7997	0.682648	-1.47110	212.568	201.629	292.983
44	53.0043	0.645745	-1.50203	210.322	206.611	294.828
45	54.2090	0.611239	-1.53215	208.109	211.435	296.672
46	55.4136	0.578984	-1.56131	205.931	215.399	298.000
47	56.6183	0.548839	-1.58957	203.789	217.426	298.000
48	57.8229	0.520670	-1.61739	201.686	219.378	298.000
49	59.0276	0.494349	-1.64469	199.622	221.257	298.000
50	60.2322	0.469754	-1.67139	197.601	223.065	298.000
51	61.4369	0.446771	-1.69740	195.621	224.803	298.000
52	62.6415	0.425290	-1.72261	193.686	226.472	298.000
53	63.8461	0.405209	-1.74692	191.797	228.074	298.000
54	65.0508	0.386431	-1.77019	189.955	229.611	298.000
55	66.2554	0.368865	-1.79227	188.161	231.084	298.000
56	67.4601	0.352427	-1.81298	186.417	232.493	298.000
57	68.6647	0.337035	-1.83213	184.726	233.839	298.000
58	69.8694	0.322617	-1.84947	183.090	235.122	298.000
59	71.0740	0.309100	-1.86471	181.513	236.341	298.000
60	72.2787	0.296422	-1.87751	180.000	237.495	298.000
61	73.4833	0.284521	-1.88744	178.559	238.580	298.000
62	74.6879	0.273341	-1.89396	177.202	239.590	298.000
63	75.8926	0.262829	-1.89639	175.945	240.515	298.000
64	77.0972	0.252937	-1.89387	174.816	241.337	298.000
65	78.3019	0.243619	-1.88521	173.863	242.024	298.000
66	79.5065	0.234833	-1.86884	173.175	242.517	298.000
67	80.7112	0.226541	-1.84251	172.945	242.681	298.000
68	81.9158	0.218707	-1.80296	173.690	242.149	298.000
69	83.1205	0.211297	-1.74546	177.812	239.138	298.000
70	84.3251	0.186052	-1.66494	185.471	233.248	298.000
71	85.5297	0.132826	-1.55204	189.782	227.469	296.242
72	86.7344	0.0824872	-1.37634	188.435	226.289	294.473
73	87.9390	0.0352501	-1.04070	183.142	228.318	292.694
74	89.1437	0.00180299	-0.10000	172.674	-3.87951	172.63
75	90.3483	0	-0.10000	163.020	-4.14655	162.967

Table 2: Values of the point-like masses $M_{[i]}$, functional parameter $\alpha_{\text{eff}[i]}$, and velocity contributions $V_{N[i]}$ and $V_{\alpha[i]}$ to $V_{\text{orb}[i]}$.

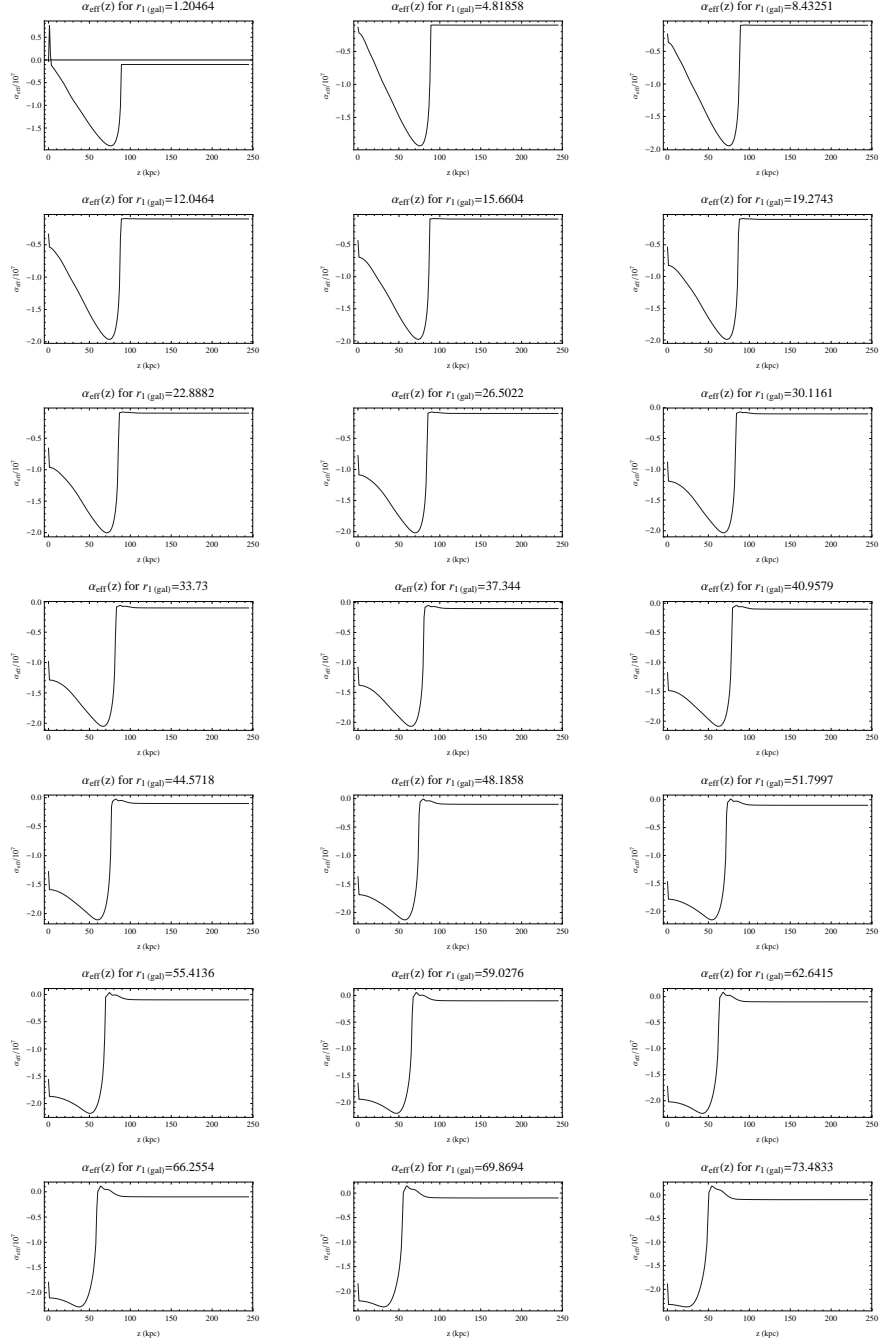


Figure 5: Estimates for α_{eff} outside of the galaxy plane assuming spherical symmetry of the functional parameters α_n 's ($\xi_\theta = 0$).

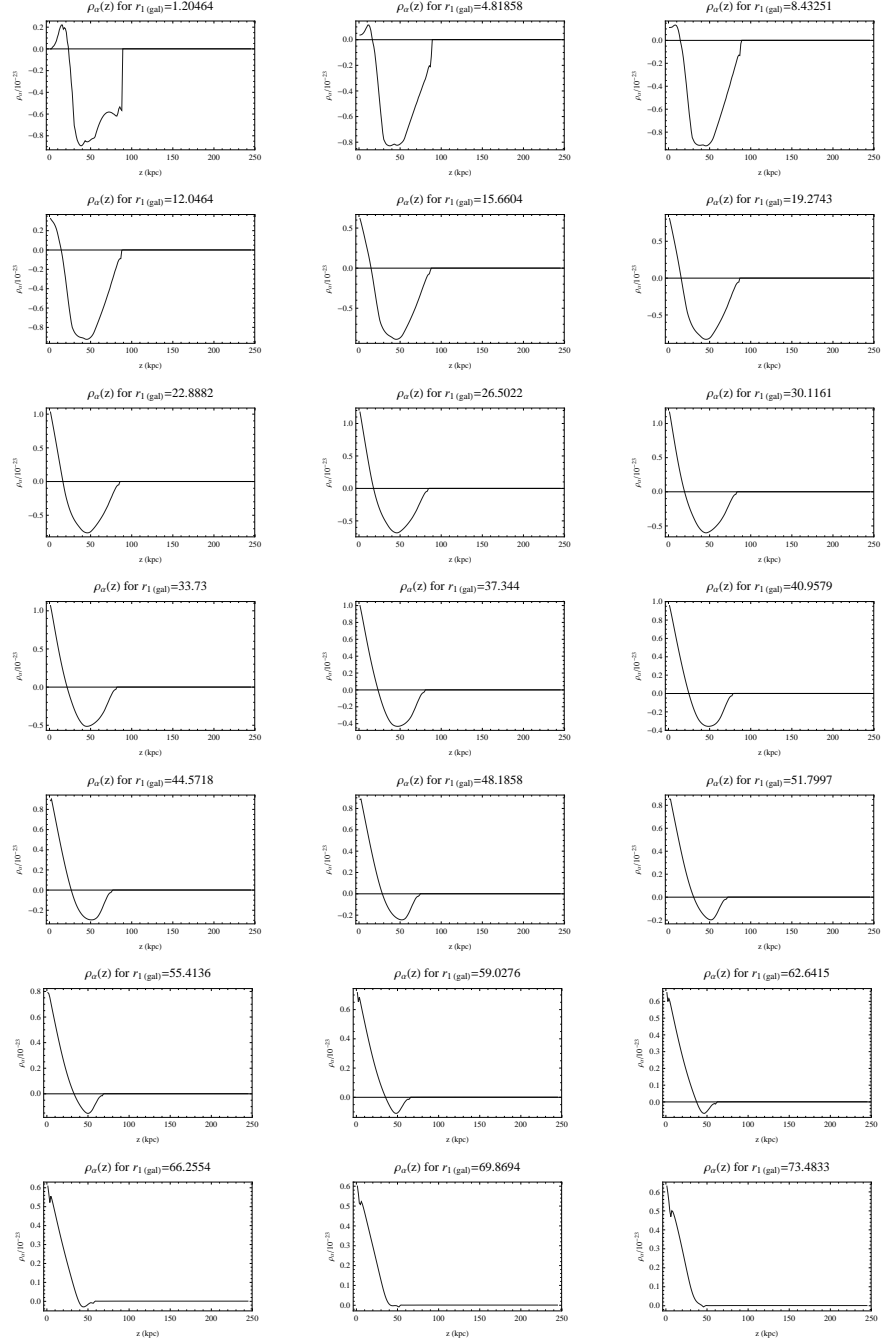


Figure 6: Estimates for ρ_α outside of the galaxy plane assuming spherical symmetry of the functional parameters α_n 's ($\xi_\theta = 0$).

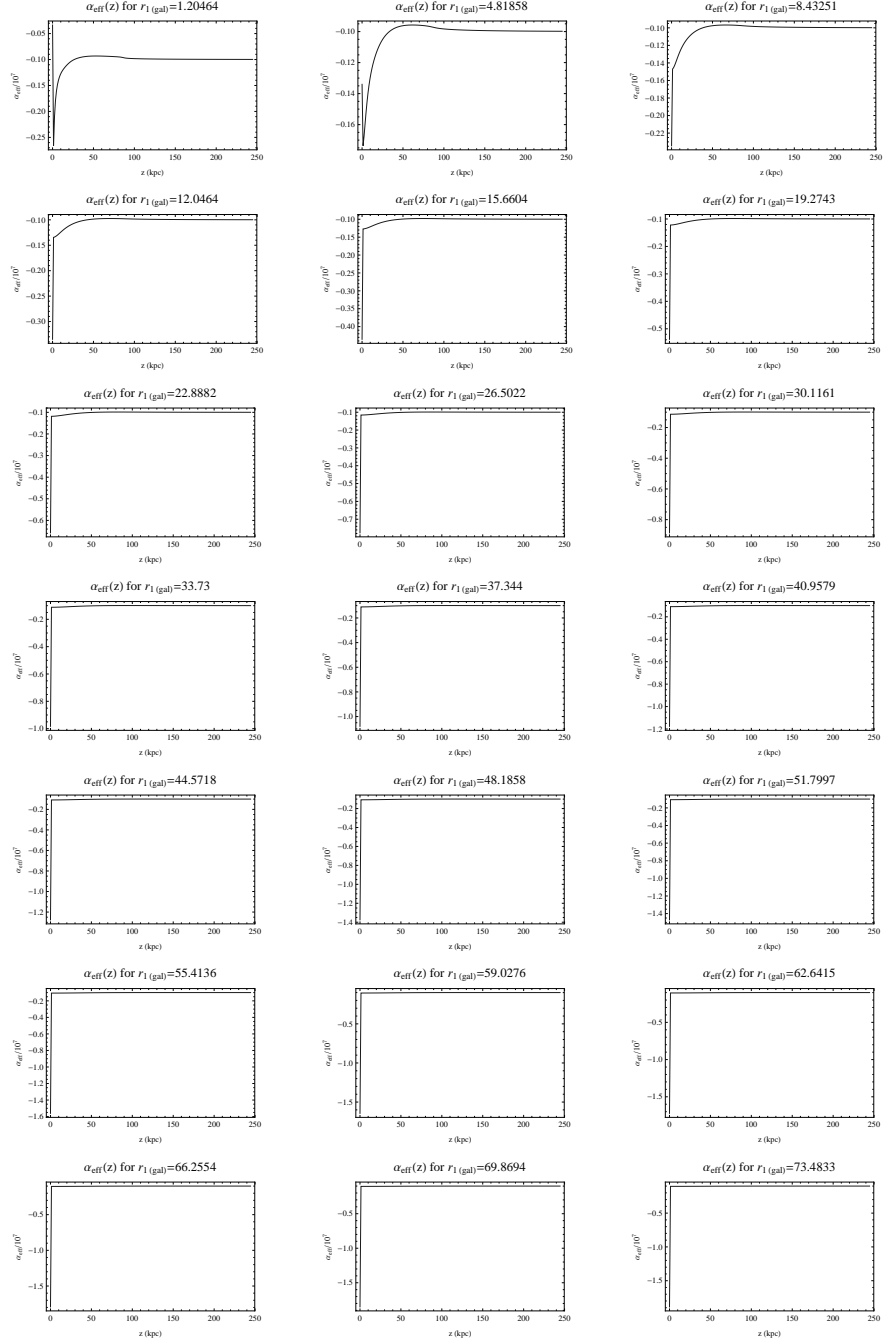


Figure 7: Estimates for α_{eff} outside of the galaxy plane assuming anisotropy of the functional parameters α_n 's ($\xi_\theta > 50000$).

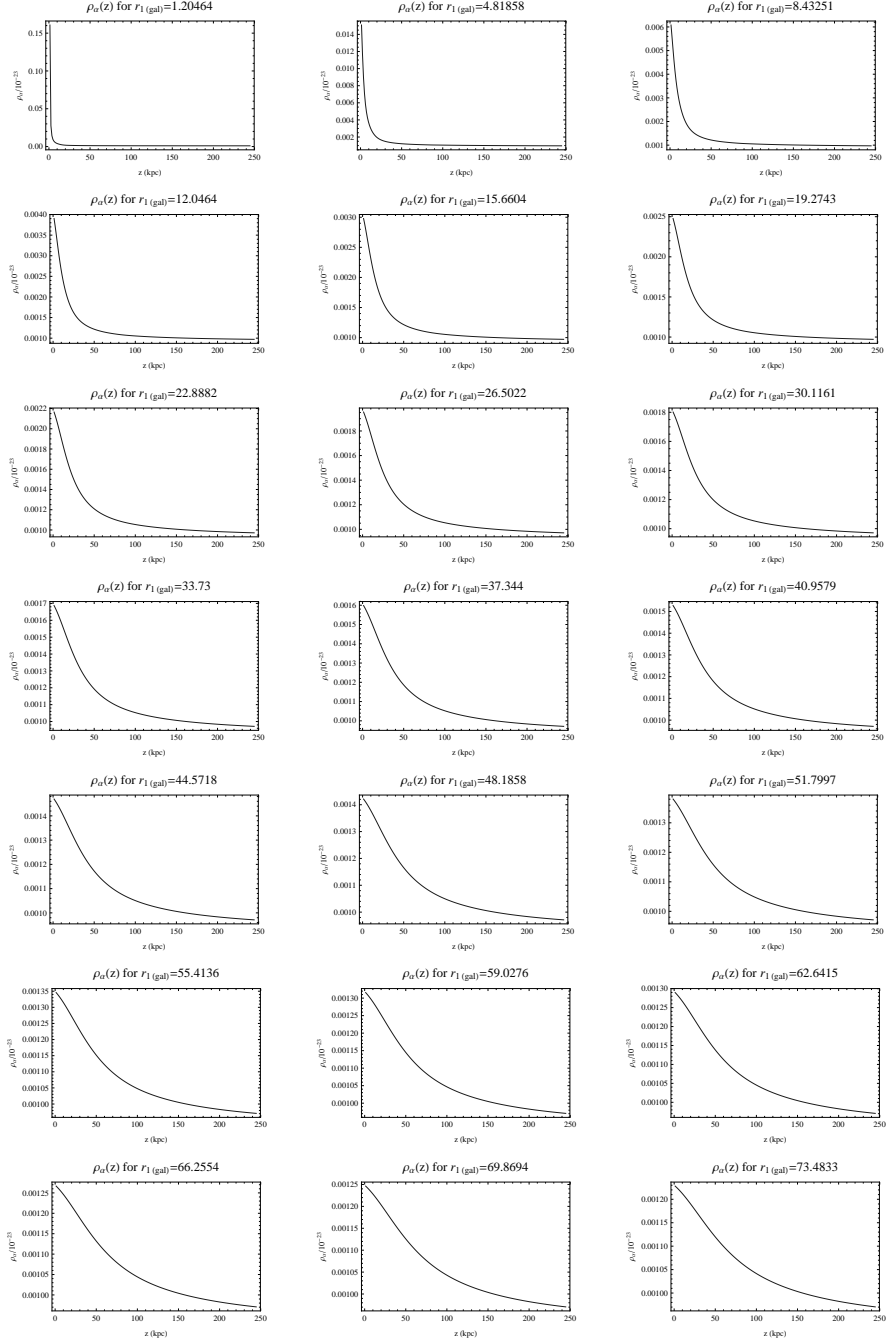


Figure 8: Estimates for ρ_α outside of the galaxy plane assuming anisotropy of the functional parameters α_n 's ($\xi_\theta > 50000$).